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Q.No → If T is an arbitrary operator on a finite dimensional non-zero space then the eigenvalues of T constitute a non-empty finite subset of the complex plane. Furthermore the number of points in this set does not exceed the dimension n of the space H. Prove it.

Proof :- We note that λ is an eigenvalue of T.

$\Leftrightarrow$ there exists a non-zero vector x such that

$(T-\lambda I)x = 0$.

$\Leftrightarrow$ $T-\lambda I$ is singular

$\Leftrightarrow$ $\det(T-\lambda I) = 0$ where "det" denotes the determinant of the matrix $T-\lambda I$.

Thus the eigenvalues of T are precisely the distinct roots of equation.

$$\det(T-\lambda I) = 0 \qquad (1)$$

which is called the characteristic equation of T.

If $[\alpha_{ij}]$ be the matrix of T relative to an ordered basis B for H, the characteristic equation can be written in the form,

$$\begin{vmatrix} \alpha_{11}-\lambda & \alpha_{12} & \cdots & \alpha_{1n} \\ \alpha_{21} & \alpha_{22}-\lambda & \cdots & \alpha_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \alpha_{n1} & \alpha_{n2} & \cdots & \alpha_{nn}-\lambda \end{vmatrix} = 0 \qquad (2)$$

This is a Polynomial equation with Complex Coefficients, of degree n in the Complex variable λ. Such an equation has exactly n Complex roots (i.e. eigenvalues of T) Some of which may be repeated, in Which Case there are less than n distinct root.

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QNo→ Prove that if T is a normal operator on H, then x is an eigenvector of T with eigenvalue $\lambda \Leftrightarrow x$ is an eigenvector of T^* with eigenvalue $\bar{\lambda}$.

Proof:- Since T is a normal operator on H, the operator $T-\lambda I$ is also normal for any Scalar λ. Now, we know that an operator A on H is normal

$$\Leftrightarrow \|A^* x\| = \|Ax\| \text{ for every } x \in H.$$

$\therefore \|(T-\lambda I)^* x\| = \|(T-\lambda I)x\|$ for every $x \in H$.

$\therefore \|T^* x - (\lambda I)^* x\| = \|Tx - \lambda x\|$ for every $x \in H$.

$\therefore \|T^* x - \bar{\lambda} x\| = \|Tx - \lambda x\|$ for every $x \in H$.

Thus, $Tx = \lambda x \Leftrightarrow T^* x = \bar{\lambda} x$.

$\therefore x$ is an eigenvector of T with eigenvalue λ.

$\Leftrightarrow x$ is an eigenvector of T^* with eigenvalue $\bar{\lambda}$.

Lemma QN.> If T is a normal operator on H then its eigen-spaces M_i's corresponding to eigenvalues λ_i's are pairwise orthogonal.

Proof:- Let $x_i \in M_i$ and $x_j \in M_j$ with $i \neq j$. Then $Tx_i = \lambda_i x_i$ & $Tx_j = \lambda_j x_j$. Hence, by lemma A, $T^* x_j = \overline{\lambda_j} x_j$.

Now, $\lambda_i(x_i, x_j) = (\lambda_i x_i, x_j) = (Tx_i, x_j) = (x_i, T^* x_j)$
$= (x_i, \overline{\lambda_j} x_j) = \lambda_j (x_i, x_j)$.
$\therefore (\lambda_i - \lambda_j)(x_i, x_j) = 0$. Since, $\lambda_i \neq \lambda_j$,
therefore, $(x_i, x_j) = 0$ i.e. $x_i \perp x_j$. Therefore $M_i \perp M_j$ for $i \neq j$.

Lemma:- If T is a normal operator on H, then its eigen-spaces M_i's corresponding to eigenvalues λ_i's then each M_i reduces T.
or, QN.> Each M_i's reduces T.

Proof:- Let $x_i \in M_i$. Then $Tx_i = \lambda_i x_i \in M_i$.
Thus, M_i is invariant under T.
Also by lemma A, $T^* x_i = \overline{\lambda_i} x_i \in M_i$.
Thus M_i is also invariant under T^*.
Therefore, each M_i reduces T.

Lemma:- If T is a normal operator on H, then eigen spaces M_i's span H.
or, QN.> The M_i's span H.

Proof:- By lemma 8, M_i's are Pairwise orthogonal. Then $M = M_1 + M_2 + \cdots + M_m$, is a closed linear subspace of H and its associated Projection is,

$$P = P_1 + P_2 + \cdots + P_m.$$

Where $P_1, P_2, \ldots, P_m$ are Projections on closed linear subspaces $M_1, M_2, \ldots, M_m$ of H.

Since by lemma each M_i reduces T, $TP_i = P_i T$ for each P_i. Therefore, $TP = T(P_1 + P_2 + \cdots + P_m)$
$= TP_1 + TP_2 + \cdots + TP_m = P_1 T + P_2 T + \cdots + P_m T$
$= (P_1 + P_2 + \cdots + P_m)T = PT$, hence M reduces T and therefore $M^\perp$ is invariant under T. If $M^\perp \neq \{0\}$, then since all the eigenvectors of T are contained in M, the restriction of T to $M^\perp$ is an operator (on a non-trivial finite dimensional Hilbert space $M^\perp$) which has no eigenvectors, and hence no eigenvalues which is impossible.

Therefore, $M^\perp = \{0\}$ and so $M = H$. Thus $H = M_1 + M_2 + \cdots + M_m$. Therefore, M_i's span H.

We find that if T is a normal operator on H, then eigenspaces M_i's are Pairwise orthogonal and span H. Thus (III) $\Rightarrow$ (I).